Diskussionsproblem. 28/2.

1. Alla olikheter är korrekta.

(a) Notera att $0 \le (\sqrt{a}-\sqrt{b})^2 = a+b-2\sqrt{ab}$. Då

$$\frac{a+b}{2} - \sqrt{ab} = \frac{(\sqrt{a}-\sqrt{b})^2}{2} = \frac{\left((\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})\right)^2}{2(\sqrt{a}+\sqrt{b})^2}$$

$$= \frac{(a-b)^2}{2}\,\frac{1}{(\sqrt{a}+\sqrt{b})^2} \begin{cases} \le \dfrac{(a-b)^2}{2}\cdot\dfrac{1}{(2\sqrt{b})^2} = \dfrac{(a-b)^2}{8b} \\ \ge \dfrac{(a-b)^2}{2}\,\dfrac{1}{(2\sqrt{a})^2} = \dfrac{(a-b)^2}{8a} \end{cases}$$

ty $0 < b \le a$.

(b) Låt $a_{n+1} = 1-(a_1+\dots+a_n) \iff a_1+\dots+a_n = 1-a_{n+1}$

$$\Rightarrow VL = \frac{a_1\cdots a_n(1-(a_1+\dots+a_n))}{(a_1+\dots+a_n)(1-a_1)\cdots(1-a_n)} = \frac{a_1}{1-a_1}\frac{a_2}{1-a_2}\cdots\frac{a_{n+1}}{1-a_{n+1}}$$

$$= \left\{ \begin{array}{l} 1-a_i = a_1+\dots+a_{i-1}+a_i+\dots+a_{n+1} = \{AM-GM\} \\ \ge n\,(a_1\cdots a_{i-1}a_i\cdots a_{n+1})^{1/n},\ \text{för } i=1,\dots,n+1 \end{array} \right\}$$

$$\le a_1\cdots a_{n+1}\cdot \underbrace{\frac{1}{n(a_2\cdots a_{n+1})^{1/n}}\cdots\frac{1}{n(a_1\cdots a_n)^{1/n}}}_{n+1 \text{ faktorer}}$$

$$= \frac{1}{n^{n+1}}\,\frac{a_1\cdots a_{n+1}}{(a_1^n a_2^n\cdots a_{n+1}^n)^{1/n}} = \frac{1}{n^{n+1}}.$$

(1)

(c) $$VL = \frac{(a+1)(b+1)(c+1)-2}{(a+1)(b+1)(c+1)}$$

$$= 1 - \frac{2}{(a+1)(b+1)(c+1)}$$

Vi vill visa att $1-\dfrac{2}{(a+1)(b+1)(c+1)} \ge \dfrac{3}{4}$

$\iff (a+1)(b+1)(c+1) \ge 8$

Denna gäller på grund av att

$\dfrac{a+1}{2} \ge \sqrt{a}\sqrt{1} = \sqrt{a},\ \dfrac{b+1}{2} \ge \sqrt{b},\ \dfrac{c+1}{2} \ge \sqrt{c}$ {AM-GM}

och $abc = 1$.

(d) $$\left|\frac{a-b}{a+b} + \frac{b-c}{b+c} + \frac{c-a}{c+a}\right| = \left|\frac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}\right|$$

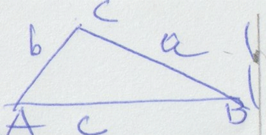
$\le$ {triangelolikheter $a<b+c,\ b<c+a,\ c<a+b$}

$$< \frac{abc}{(a+b)(b+c)(c+a)} = \left\{ \begin{array}{l} a+b \ge 2\sqrt{ab} \\ b+c \ge 2\sqrt{bc} \\ c+a \ge 2\sqrt{ac} \end{array} \right\}$$

$$\le \frac{abc}{2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ac}} = \frac{abc}{8abc} = \frac{1}{8}.$$

(2)

(e) $\sin^2\frac{A}{2} = \frac{1-\cos A}{2} = \left\{ \begin{array}{l} \text{cosinusatsen} \\ a^2 = b^2+c^2-2bc\cos A \end{array} \right.$



$$= \frac{1-\frac{b^2+c^2-a^2}{2bc}}{2} = \frac{a^2-(b-c)^2}{4bc}$$

$$= \frac{(a-b+c)(a+b-c)}{4bc}$$

P.S.S.

$$\sin^2\frac{B}{2} = \frac{(b-c+a)(b+c-a)}{4ac}$$

$$\sin^2\frac{C}{2} = \frac{(c-a+b)(c+a-b)}{4ab}$$

$$\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2} = \frac{(a-b+c)^2(a+b-c)^2(b+c-a)^2}{4^3(abc)^2}$$

$$\Rightarrow \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{(a-b+c)(a+b-c)(b+c-a)}{8\cdot abc}$$

$$= \left\{ \begin{array}{l} \text{Ravis substitution} \\ a=x+y,\ b=y+z,\ c=z+x \end{array} \right\}$$

$$= \frac{xyz}{(x+y)(y+z)(z+x)} = \left\{ \begin{array}{l} x+y \geq 2\sqrt{xy} \\ y+z \geq 2\sqrt{yz} \\ z+x \geq 2\sqrt{zx} \end{array} \right\}$$

$$\leq \frac{xyz}{8xyz} = \frac{1}{8}.$$

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