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On the Rational Values of Trigonometric Functions of Angles that Are Rational in Degrees

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The concept of commensurable magnitudes can be traced back to the ancient Greeks [7]. Recall that two magnitudes are commensurable if their ratio is given by a pair of positive integers. An interesting theorem in number theory related to commensurable angles states:

Theorem 1. *If α is rational in degrees, say $\alpha = r360°$ for some rational number r, then the only rational values of the trigonometric functions of α are as follows:*

$$\cos(\alpha),\ \sin(\alpha) = 0,\ \pm(1/2),\ \pm 1,$$
$$\sec(\alpha),\ \csc(\alpha) = \pm 1,\ \pm 2,$$
$$\tan(\alpha),\ \cot(\alpha) = 0,\ \pm 1.$$

This result is sometimes referred to as Niven's theorem, as it appears in two of his books (see [12, Corollary 3.12] or [13, Theorem 6.16]). On the other hand, Niven himself has written:

> A proof of [Theorem 1] . . . was given by J. M. H. Olmsted [14]. The topic is a recurring one in the popular literature: as examples we cite H.A. Bradford and H. Eves [1]; R. W. Hamming [8]; E. Swift [16]; R. S. Underwood [17].

In the last fifty years, other authors have produced proofs of this result, and more recently many applications of it, both elementary and advanced, have arisen [2–6, 9–11, 13, 15].

We present a new proof that is very elementary. In our opinion, it is feasible for teachers and it is has proven easy to understand for high school students.

Proof of the theorem

Our proof avoids the advanced tools like induction, the de Moivre formulas, Chebyshev polynomials, or cyclotomic polynomials that are typically used. For instance, see Bergen [3], Calcut [6], Niven, Zuckerman, and Motgomery [13], or Underwood [17]. In Jahnel [11], the reader can find interesting insights for an elementary proof, even if some infinite processes are used in the argument.

The idea of our proof is essentially based on the periodicity of the function $\cos(x)$, which allows us to reduce the problem to the analysis of a few cases.

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Proof. It is sufficient to confine the proof to the cosine and tangent functions.

First, we assume that $\cos(\alpha)$ is a rational number different from 0. We will prove that

$$\cos(\alpha) \in \left\{\pm 1,\ \pm\frac{1}{2}\right\}.$$

Write

$$\cos(\alpha) = \frac{p}{q} \neq 0,$$

where $p, q \in \mathbb{Z}$ and are coprime.

Fix n such that $n\alpha$ is an integer multiple of $360°$.

For every positive integer i, we put $\alpha_i = (\frac{i}{n})360°$. Let m be such that $\alpha_m = \alpha$, so that by hypothesis $\cos(\alpha_m)$ is a rational number. It follows that

$$T_n := \{\cos(\alpha_i) \in \mathbb{Q} \setminus \{0\} \mid i \in \mathbb{N}\},$$

is a finite, non-empty set of rational numbers. Each element of T_n is of the type $\cos(\alpha_i) := \frac{p_i}{q_i}$, where $p_i \in \mathbb{Z} \setminus \{0\}$ and $q_i \in \mathbb{N}$ are coprime. Among them we may choose an element, say $\cos(\alpha_k) = \frac{p_k}{q_k}$, whose denominator q_k is the greatest.

We now have that

$$\alpha_{2k} = 2\alpha_k = \left(2\,\frac{k}{n}\right) 360°.$$

It follows that

$$\cos(\alpha_{2k}) = \cos(2\alpha_k) = \frac{2p_k^2 - q_k^2}{q_k^2}$$

is a rational number.

Note that p_k and q_k are coprime. It follows that if q_k is odd, then $2p_k^2 - q_k^2$ and q_k^2 are both different from 0 and coprime. Hence, by the choice of q_k, we have that $q_k \geq q_k^2$. This yields $q_k = 1$.

If $q_k = 2s$ is even ($s \in \mathbb{N}$), then

$$\cos(2\alpha_k) = \frac{2p_k^2 - q_k^2}{q_k^2} = \frac{p_k^2 - 2s^2}{2s^2}.$$

On the other hand, p_k and q_k are coprime, and each divisor of s is also a divisor of q_k. Therefore, the last term above is a reduced fraction different from 0. By the choice of q_k, we have that

$$q_k \geq 2s^2 = \frac{q_k^2}{2}.$$

Thus, q_k is either 1 or 2, and hence $\cos(\alpha) \in \left\{\pm 1,\ \pm\frac{1}{2}\right\}$.

The $\tan(\alpha)$ part is routine, and we will only sketch the proof.

Suppose that $\tan(\alpha)$ is a rational number. Clearly, we may assume that $0 \leq \alpha < 180°$. It follows by a basic trigonometric relation that

$$\cos(2\alpha) = \frac{1 - \tan^2(\alpha)}{1 + \tan^2(\alpha)},$$

is a rational number. The first part of the proof shows that

$$2\alpha \in \{0^\circ, 60^\circ, 90^\circ, 120^\circ, 180^\circ, 240^\circ, 270^\circ, 300^\circ\}.$$

It follows that

$$\alpha \in \{0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ, 120^\circ, 135^\circ, 150^\circ\}.$$

By hypothesis $\tan(\alpha)$ is rational, so that the only possible values for α are 0°, 45°, 135°.

This completes the proof. ■

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Summary. We give an elementary proof of the well-known theorem that gives the rational values of trigonometric functions of angles that are rational in degrees. Our proof avoids the traditional arguments based on induction, the de Moivre formulas, Chebyshev polynomials, or cyclotomic polynomials that are typically used.

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